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Application of Laguerre functions to data compression

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Abstrakt This article deals with the use of the Laguerre functions in the data compression. After the short introduction the definition of the Laguerre polynomials and functions is given. The application of the discrete Laguerre transform on the data compression is shown. It is pointed out that the discrete Laguerre transform can give better results than the discrete cosine transform in the task of the data compression.

Klíčová slova Laguerre, compression, polynomial

1. LAGUERRE POLYNOMIALS AND FUNCTIONS

The orthogonal functions are used in many fields. In this work we will focus on the use of the orthogonal Laguerre functions in the data compression task. The Laguerre functions were introduced by Edmond Laguerre more than 150 years ago. Many applications of them on various problems in mathematics, physics and electrotechnics were found. In this paper there will be shown some examples how can the usage of the Laguerre functions help to get better results in the data compression task. In the following some basic definitions will be presented. The generalized Laguerre polynomials $l_n^a(t)$ are the solutions of the following differential equation

$$ty'' + (a+1-t)y' + ny = 0, n \in \mathbb{N}_0, a \in (-1, \infty).$$

The above differential equation can be converted into the Sturm-Liouville form by multiplying $t^a e^{-t}$

$$-\frac{d}{dt}(t^{a+1}e^{-t}y') = nt^a e^{-t}y, n \in \mathbb{N}_0, a \in (-1, \infty).$$

So, the vast theory about the Sturm-Liouville systems (see [5]) can be used for analyzing the properties of the solutions of the above equation, i.e. for analyzing the generalized Laguerre polynomials. The generalized Laguerre polynomials can be written in the following form

$$l_n^{(a)}(t) = \frac{t^{-a}e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^{n+a}).$$

The generalized Laguerre polynomials form the complete orthogonal system in $L_2(0, \infty)$ with respect to the weight function $t^a e^{-t}$, i.e.,

$$\int_0^\infty l_i^{(a)}(t) l_j^{(a)}(t) t^a e^{-t} dt = \binom{n+a}{n}\Gamma(a+1)\delta_{i,j}.$$

One of the most important properties of the orthogonal polynomials is that they satisfy the 3-term recurrence relation. The generalized Laguerre polynomials satisfy following relation

$$(n+1)l_{n+1}^a(t) = (2n+1+a-t)l_n^a(t) - (n+a)l_{n-1}^a(t).$$

The above relation is very important for practical computation of the generalized Laguerre functions on the computer. The so-called simple Laguerre polynomials can be found in the literature. These can be obtained simply by putting $a = 0$,

$$l_n^{(0)}(t) = l_n(t).$$

The orthonormalized Laguerre polynomials are called the Laguerre functions $L_n^{(a)}(t)$,

$$L_n^{(a)}(t) = \sqrt{\frac{n!\,t^a}{\Gamma(n+a)e^t}}\, l_n^{(a)}(t).$$

The special case for $a = 0$ is $L_n^{(0)}(t) = e^{-\frac{t}{2}} l_n^{(0)}(t)$. These simple Laguerre functions are used in the experiments in the chapter 3.

There are many reference books and articles about the theory of Laguerre polynomials and functions. Their main application is in the field of modelling the dynamical systems. In [10] there is the simple example of the black box model identifying based on the transform to the Laguerre functions basis. However, it's possible that their potential for practical computation wasn't fully exploited yet due to the numerical problems, which occurred during their implementation.

As pointed out in the fairly new article [11] the source of the numerical problems is mainly in the usage of Laguerre polynomials instead of the Laguerre functions. The Laguerre polynomials are usable only inside small intervals due to their extremely ill-conditioned behavior. The basis of the Laguerre functions for the discrete Laguerre transform will be used in the following chapter.

The application of the Laguerre function in the data compression task will be presented. The up to present research in these two tasks will be summarized. After the introduction some possible ways how to continue the research in these two fields will be presented.

2. DLT FOR DATA COMPRESSION

In this section the comparison between the discrete Laguerre and cosine transforms (DLT, DCT) when applied on the data compression task will be presented.

The DCT was introduced in 1974 into electrical engineering literature by N. Ahmed, T. Natarajan and K.R. Rao in their article [3]. It is the real version of the discrete Fourier transform. Nowadays DCT and its modifications like the modified discrete cosine transform are the cores of many algorithms for data compression and signal processing. For example, DCT is used in the JPG and MP3 algorithms for image and sound processing.

The main idea behind the use of the orthogonal transforms for data compression is their so-called "energy compaction property", see [4]. It means that the most of the information is stored in the first few Fourier coefficients of the Fourier series for the original data.

Although there are many articles about the Laguerre polynomials and functions, the transform similar to DCT based on the Laguerre orthonormal functions wasn't introduced till 1995 when the article [2] appeared. In that article the DLT was defined with the help of Gauss-Laguerre integration in the similar way as the other finite orthonormal transforms. It was suggested, that this transform could lead to the better results in the data compression tasks than the DCT. It means that the DLT have the same "energy compaction property" as the DCT. This will work especially for the vectors, that decay exponentially to zero, i.e., that have the similar behavior as the Laguerre basis functions.

Since 1995 the DLT was used in the modelling only few times. The article [1] was published in 1995 after the original article about DLT. In [1] there was shown the application of the DLT to the speech coding. The DLT was compared to DCT in the classic speech coding algorithm [9]. It was shown, that it outperforms the DCT at low bitrates.

In 2000 and 2001 the DLT was applied to the digital image watermarking by M.S.A. Gilani and A.N. Skodras in their articles [6], [7] and [8]. It was shown that the image quality is better with the use of the DLT instead of the classical approach with the DCT.

3. EXAMPLES OF DATA COMPRESSION

Now the following data compression task for $z \in \mathbb{R}^N$ will be presented. Let's consider the Fourier expansion for the vector z, i.e.,

$$z = \sum_{i=1}^{N} c_i u_i,$$

where $\{c_i\}$ are the Fourier coefficients for some orthonormal basis $\{u_i\}$ of $\mathbb{R}^N$. Now consider the truncated expansion for some $K \leq N, K \in \mathbb{N}$, i.e.,

$$w = \sum_{i=1}^{K} c_i u_i.$$

The vector reconstruction w is the approximation of the vector z. This move from the vector z to the vector w is often called the compression of the vector z or simply the reduction of the model. The main idea of this compression is that the most of the information in the vector is contained in the first few Fourier coefficients of the vector expansion, i.e. that the used orthogonal transform has the "energy compaction property".

In the following there are the pictures of the vector of length $N = 28$ reconstruction for $K = 4,8,24$. The graphs and tables of the relative compression error (RCE) $\| z - w \| / \| z \|$ are shown for $K = 4,8,12,16,20,24$. All the experiments were done in MATLAB.

The 1.vector is the exponentially damped sinusoid sampled in the interval $[0, 2.8]$, i.e.,

$$z_i = e^{-0.2i} \sin(i), i = 1..28.$$

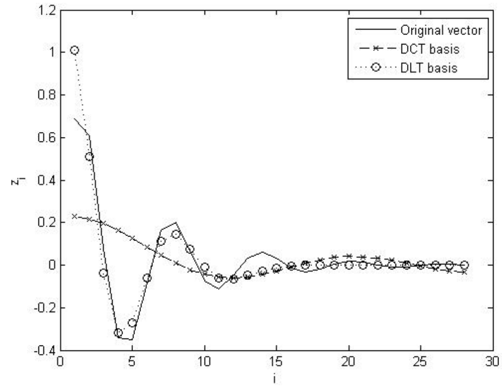

Figure 1: Reconstruction of the 1.vector for $K = 4$

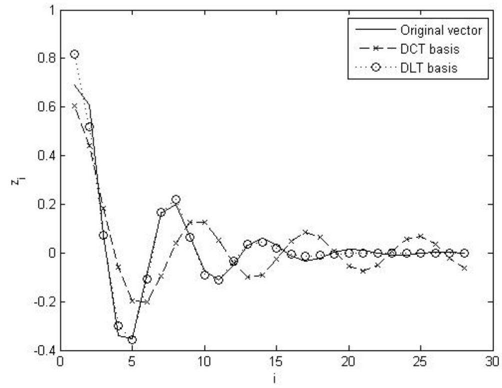

Figure 2: Reconstruction of the 1.vector for $K = 8$

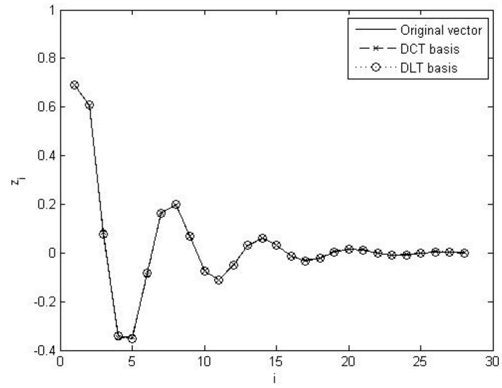

Figure 3: Reconstruction of the 1.vector for $K = 24$

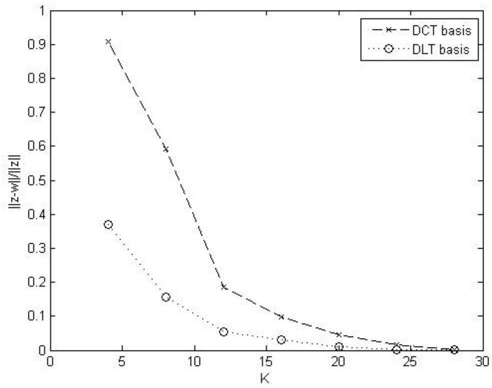

Figure 4: RCE $\| z - w \| / \| z \|$ of the 1.vector

K	4	8	12	16	20	24
DCT	0.906	0.591	0.186	0.096	0.045	0.015
DLT	0.370	0.157	0.055	0.029	0.008	0.001

Table 1: RCE $\| z - w \| / \| z \|$ for DCT and DLT basis of the 1.vector

The 2. vector is the unit descent function sampled in the interval [0,2.8], i.e.,

$$z_i = \begin{cases} 1, & 1 < i \leq 10, \\ 0, & 11 \leq i \leq 28. \end{cases}$$

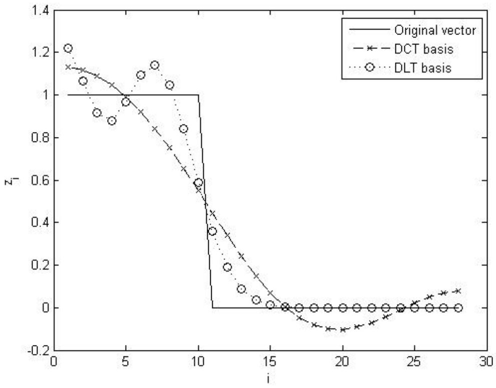

Figure 5: Reconstruction of the 2.vector for $K = 4$

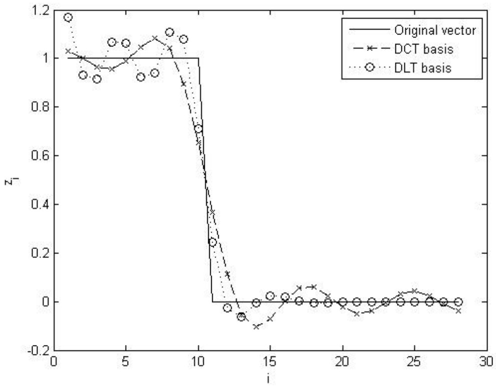

Figure 6: Reconstruction of the 2.vector for $K = 8$

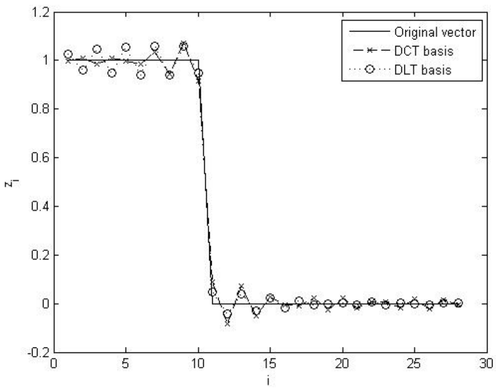

Figure 7: Reconstruction of the 2.vector for $K = 24$

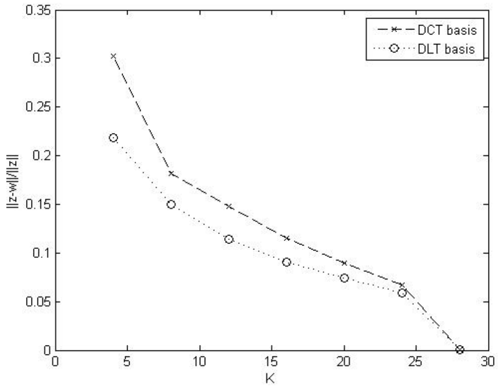

Figure 8: RCE $\| z - w \| / \| z \|$ of the 2.vector

K	4	8	12	16	20	24
DCT	0.302	0.182	0.148	0.115	0.090	0.067
DLT	0.218	0.150	0.114	0.090	0.074	0.059

Table 2: RCE $\| z - w \| / \| z \|$ for DCT and DLT basis of the 2.vector

4. CONCLUSION

In the examples in the previous chapter it was presented that the DLT performs significantly better than the DCT in the term of the relative compression error. Thought it was pointed out in [2] that it would be possible to obtain such good results in the data compression task there are still many open questions and the future research in this field can bring some interesting facts. The future work will be focused on the searching for the precise definition of the classes of functions for which the use of DLT can bring such good results and on the computational aspects of DLT. Also the use of the generalized Laguerre functions with different values of the parameter a and their comparison to the simple Laguerre functions is interesting and it could lead to even better results.

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