GRANT journal
ISSN 1805-062X, 1805-0638 (online), ETTN 072-11-00002-09-4

EUROPEAN GRANT PROJECTS | RESULTS | RESEARCH & DEVELOPMENT | SCIENCE

Transverse Vibration of the Simply Supported Beam Loaded by Pedestrians

Vladimír Šána[1]
Michal Polák[2]

[1] Czech Technical University in Prague; Faculty of Civil Engineering; Thákurova 7, 166 29 Praha 6 – Dejvice; vladimir.sana@fsv.cvut.cz
[2] Czech Technical University in Prague; Faculty of Civil Engineering; Thákurova 7, 166 29 Praha 6 – Dejvice; polak@fsv.cvut.cz

Grant: SGS 12/117/OHK1/2T/11
Grant Name: Aplikace diagnostických metod u materiálů a konstrukcí
Oborové zaměření: JM – Inženýrské stavitelství

Abstract The presented article is focused on a dynamic analysis of the simply supported beam, which is loaded by the harmonically varying force and by the force with constant magnitude. Both of these types of loading are moving across the structure with constant velocity v_p. Results of the theoretical analysis are compared with the experimentally obtained results and with the deterministic model based on the theory of the Fourier's Series according to the *Bachmann*, *Ammann*, *Young* etc. The structure, chosen for a dynamical analysis, is the real footbridge across the Opatovicka street, placed in the Prague with a composite cross – section.

Key words forced vibration, dynamical loading of footbridges, vibration due to pedestrians

1. INTRODUCTION

Currently, the vertical loading of footbridges, caused by human activities such as walking or running is considered as the harmonically varying force applied at resonance with some natural frequency, which is placed at the point with maximal value of the vertical deflection of corresponding natural mode. In this article are presented results from the numerical analysis, where three alternative models of loading were considered. The first model is the deterministic model, which is described by the equation

$$F(t) = m_p \cdot g\left[1 + \alpha_1 \sin\left(2\pi f_p t\right)\right] \qquad (1)$$

where m_p is the body weight of pedestrians, g is the gravitational acceleration, α_1 is the coefficient of the Fourier's Series and f_p is the pacing frequency. The coefficients α_1 determined by different authors are summarized in [5]. The comfort criteria for pedestrians during walking along the structure are expressed by the maximal vertical, respective lateral, value of acceleration. Therefore the acceleration of vibration were measured and computed.

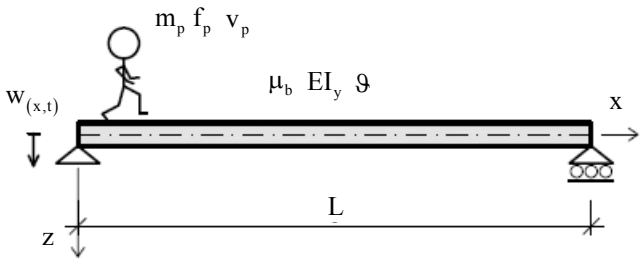

Fig 1 The static scheme of the solved structure

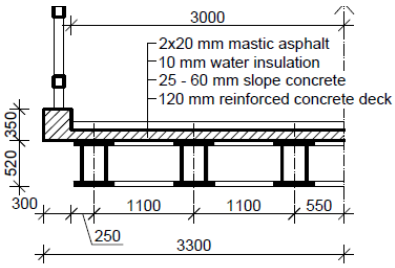

Fig 2 The cross-section of the solved structure

The second model is the force with a constant magnitude moving along the structure with constant velocity. It is described by the equation

$$F = m_p \cdot g + \left(\frac{1}{\tau} m_p \sqrt{2gh}\right) \qquad (2)$$

τ is the time of contact between pedestrian's foot and the bridge deck and h is the height of free fall.

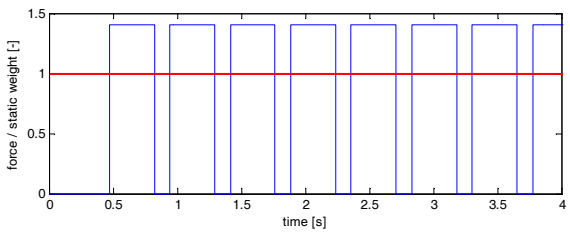

Fig 3 The time behaviour of force with constant magnitude

The third model, which has been considered, is the harmonic force (deterministic model of pedestrian), moving along the structure with constant velocity and described also by the relation (1).

2. MATHEMATICAL DESCRIPTION

The footbridge structure is modeled by a discrete system of N mass points with N degrees of freedom (the MDOF system) see *Fig. 4*. In described study, only the vertical DOFs are considered. Hence the stiffness matrix $\mathbf{K}$ has to be reduced by the static condensation process to the matrix $\mathbf{K}^{\mathbf{red}}$ where the massless DOFs, corresponding to rotational DOFs, are eliminated. Thus $\mathbf{K}^{\mathrm{red}} = \mathbf{K}_{aa} - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\mathbf{K}_{ba}$. The mass matrix $\mathbf{M}$ is considered as diagonal with elements $\mu L/(N+1)$ at the principal diagonal. The dimension of these matrices is $\langle N \times N \rangle$. For assembling the damping matrix $\mathbf{C}$, the model of Rayleigh damping is used. In according to this assumption is the matrix $\mathbf{C}$ expressed by a linear combination of the matrices $\mathbf{M}$ and $\mathbf{K}$ then $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$. The coefficients of linear combination are formulated as $\alpha = \xi_1\omega_1$ $\beta = \xi_1/\omega_1$ where $\omega_1 = 2\pi f_{(1)}$, $f_{(1)}$ is the first natural bending frequency. Then the problem of forced vibration should be written in matrix form (3).

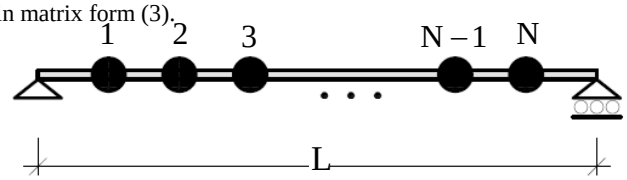

Fig 4 The discretized MDOF system

$$[\mathbf{M}]\{\ddot{\mathbf{w}}\} + [\mathbf{C}]\{\dot{\mathbf{w}}\} + \left[\mathbf{K}^{\mathrm{red}}\right]\{\mathbf{w}\} = \{\mathbf{F}\} \tag{3}$$

where $\{\ddot{\mathbf{w}}\}\{\dot{\mathbf{w}}\}\{\mathbf{w}\}$ are column vectors of acceleration, velocity and deflection. Size of these vectors is $\langle N \times 1 \rangle$. The right side of the equation (3) is the column force vector with same size $\langle N \times 1 \rangle$. The process of assembling the stiffness matrix, with re-organized columns and rows for the static condensation method, is explained by following:

$$[\mathbf{K}] = \begin{array}{c} \begin{array}{cccc} \mathbf{K}_{aa} & & \mathbf{K}_{ab} & \\ w_a & w_b & \varphi_a & \varphi_b \end{array} \\ \left[\begin{array}{cc|cc} \frac{12EI}{L^3} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -\frac{12EI}{L^3} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \\ \hline \frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{4EI}{L} \end{array}\right] \\ \begin{array}{cc} \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{array} \end{array}$$

2.1 The deterministic model

The simplest model of a forced vibration due to human walking is the deterministic model, described by the relation (1). This force model is placed at the most efficient point of the adequate natural mode of vibration. If we consider the first natural mode of vibration and boundary conditions, which define a simply supported beam, the most efficient location is in the mid-span. In this paper is this model also considered as moveable force with constant velocity of walking or running. The coefficients α_1 are determined according to *Blanchard* [5], *Young* [5], *Bachmann* and *Ammann* [1].

2.2 The force with constant magnitude

This concept is derived from the assumption that human body is acting as a mass point, which is falling free down to the bridge desk. The impact velocity is $v = \sqrt{2gh}$ and the quantity of motion is expressed, according to the Newton's second law of motion, as $p = m_p\sqrt{2gh}$ divided of contact time τ, we receive the force of the impact. Time behaviour of the force is considered as periodic rectangular impulse see *Fig. 3*.

3. THE SOLUTION OF THE FORCED VIBRATION

The equation (3) is solved via vibration modes decomposition, which transforms the system of N simultaneous second-order differential equations to the N independent second-order differential equations. This advantage occurs only for standardized modes of vibration. Firstly the modal matrix $\mathbf{\Phi}$ has to be computed. In the modal matrix are the standardized modes of vibration arranged to the columns. The natural modes of vibration were calculated via the Inverse Iteration Method (or the Stodola's method) with using the Gramm – Schmidt ortogonalization. If we use a substitution $\{\mathbf{w}\} = [\mathbf{\Phi}]\{\mathbf{q}\}$ and multiply whole equation (3) with $\mathbf{\Phi}^{\mathrm{T}}$ from the left, we receive the relation

$$\begin{aligned} &[\mathbf{\Phi}]^{\mathrm{T}}[\mathbf{M}][\mathbf{\Phi}]\{\ddot{\mathbf{q}}\} + [\mathbf{\Phi}]^{\mathrm{T}}[\mathbf{C}][\mathbf{\Phi}]\{\dot{\mathbf{q}}\} + \ldots \\ &\ldots + [\mathbf{\Phi}]^{\mathrm{T}}\left[\mathbf{K}^{\mathrm{red}}\right][\mathbf{\Phi}]\{\mathbf{q}\} = [\mathbf{\Phi}]^{\mathrm{T}}\{\mathbf{F}\} \end{aligned} \tag{4}$$

The meaning of the parts in the equation (4) is explained by following relations: $[\mathbf{\Phi}]^{\mathrm{T}}[\mathbf{M}][\mathbf{\Phi}] = [\mathbf{E}]$, $[\mathbf{\Phi}]^{\mathrm{T}}\left[\mathbf{K}^{\mathrm{red}}\right][\mathbf{\Phi}] = \left[\mathbf{\Omega}^2\right]$ where $[\mathbf{E}]$ is the unit matrix and $\left[\mathbf{\Omega}^2\right]$ is the spectral matrix contains circular natural frequencies at the principal diagonal.
The modes of vibration, for simply supported beam, should be also described by continuous function $\phi^i_{(x)} = \sin(i\pi x/L)$ $i = 1,\ldots,N$. If we assume, that the force is moving along the structure with constant velocity v_p, the function $\phi^i_{(x)}$ could be transformed to the time domain via the substitution $x = v_p t$, therefore we are able to write, that $\phi^i_{(t)} = \sin(i\pi v_p t/L)$. Thus we can right side of the equation (4) in form

$$\{\mathbf{F}\} \otimes \left(1/\sqrt{\mu L/2} \cdot \sin(i\pi v_p t/L)\right) \tag{5}$$

instead of $[\mathbf{\Phi}]^{\mathrm{T}}\{\mathbf{F}\}$. The vector $\{\mathbf{F}\}$ contains amplitudes of the force, which is acting at the structure. The symbol $\otimes$ expresses multiplication of corresponding elements in vectors, multiplying with all natural modes of vibration. In the case of harmonic force, which is moving
along the structure, the right side of the equation (4) is described by the relation

$$\{\mathbf{F}\} \otimes \left(1/\sqrt{\mu L/2} \cdot \sin(i\pi v_p t/L) \cdot \sin(2\pi f_p t)\right) \tag{6}$$

After executing the modifications of the second – order differential equations (4) we are able to compute the unknown acceleration, velocity and the deflection of each discrete point. The Newmark's β integration method was used for the solution of this problem.

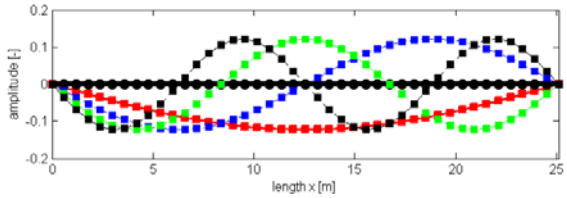

Fig 5 The first four modes of vibration of the simply supported beam for 41 DOFs

The Newmark's β integration method is described by follows relations:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \Delta t \dot{\mathbf{w}}_n + (0.5-\delta)\Delta t^2 \ddot{\mathbf{w}}_n + \delta \ddot{\mathbf{w}}_{n+1} \Delta t^2 \tag{7}$$

$$\dot{\mathbf{w}}_{n+1} = \dot{\mathbf{w}}_n + (1-\gamma)\Delta t \ddot{\mathbf{w}}_n + \gamma \Delta t \ddot{\mathbf{w}}_{n+1} \tag{8}$$

$$\ddot{\mathbf{w}}_{n+1} = \left[\mathbf{M} + \gamma \Delta t \mathbf{C} + \delta \Delta t^2 \mathbf{K}\right]^{-1} \begin{Bmatrix} \mathbf{F}_{n+1} - \mathbf{C}\left\{\dot{\mathbf{w}}_n + (1-\gamma)\Delta t \ddot{\mathbf{w}}_n\right\} \dots \\ -\mathbf{K}\left\{\mathbf{w}_n + \Delta t \dot{\mathbf{w}}_n + (0.5-\delta)\Delta t^2 \ddot{\mathbf{w}}_n\right\} \end{Bmatrix} \tag{9}$$

Note, that these equations describe the solution of general dynamic problem (3). If we use the substitution $\{\mathbf{w}\} = [\mathbf{\Phi}]\{\mathbf{q}\}$ the primary unknowns are $\{\mathbf{q}\}$ $\{\dot{\mathbf{q}}\}$ $\{\ddot{\mathbf{q}}\}$ and the meaning of matrices $[\mathbf{K}]$ $[\mathbf{C}]$ $[\mathbf{M}]$ is: $[\mathbf{K}] = [\ \ ^2]$ $[\mathbf{C}] = [\ \ ]^T [\mathbf{\Phi}][\ \ ]$ and $[\mathbf{M}] = [\mathbf{I}]$ this is the consequence of the multiplying the matrices with standardized modal matrix $[\mathbf{\Phi}]$.

3.1 The numerical values

The structure is described by following values, which were determined experimentally or have been taken from the static design.

The bending stiffness of the cross – section is considered as: $EI_y = 3.83 \cdot 10^6\ \mathrm{kNm^2}$, the continuous mass of the beam $\mu = 5.3\mathrm{t/m}$, theoretical span of the structure $L = 25.1\ \mathrm{m}$ and the logarithmical damping decrement, which have been found out experimentally as $\vartheta = 0.088$. The damping ratio ξ then could be computed with using the formulae $\xi = \vartheta / 2\pi$, thus $\alpha = 0.1851$ and $\beta = 0.0011$. In the case of loading were chosen following parameters:

The deterministic model

The body weight of the two synchronous pedestrians is $m_p = 160\ \mathrm{kg}$, the pacing frequency f_p is equal to the first natural bending frequency of the footbridge, the velocity of motion $v_p = 2.6\ \mathrm{ms^{-1}}$.

The constant force

The step length $d_p = 0.8\ \mathrm{m}$, time of contact between the bridge deck and human foot $\tau = 0.4\ \mathrm{s}$, $h = 0.1\ \mathrm{m}$ is the height of the free fall. The footfall forces with contact time are enable e.g. in [1] or in [2]

4. EXPERIMENT

The in-situ experiment was focused on the acceleration response of the footbridge across the Opatovicka Street, which was loaded by different group of synchronous pedestrians and vandals. For the comparison with the theoretical analysis, presented in this study, the response caused by two synchronous pedestrians-runners was chosen. Two runners with whole weight approximately $m_p = 160\ \mathrm{kg}$ were jogging across the footbridge with pacing frequency equal to the first natural bending frequency, which has been found out experimentally as: $f_{(1)} = 2.72\mathrm{Hz}$.

Fig 6 The placement scheme of the acceleration sensors on the bridge deck, taken from [3]

5. RESULTS

Loading		Moving deterministic model			
		Blanchard	Bachmann	Young	/
a	alpha	0.275	0.5	0.655	1.0
Maximum		0.17	0.31	0.41	0.62
Minimum		-0.17	-0.31	-0.41	-0.62

Tab 1 The summary of evaluated acceleration from theoretical analysis

Loading		Constant Force	Deterministic Model		
			Blanchard	Bachmann	Young
a	alpha		0.275	0.5	0.655
Maximum		0.41	0.23	0.39	0.51
Minimum		-0.42	-0.23	-0.39	-0.51

Tab 2 The summary of evaluated acceleration from theoretical analysis

Experimental data				
	18		19	
Measured point	2	3	2	3
Maximum	0.43	0.44	0.40	0.38
Minimum	-0.48	-0.47	-0.50	-0.55

Tab 3 The summary of experimental results

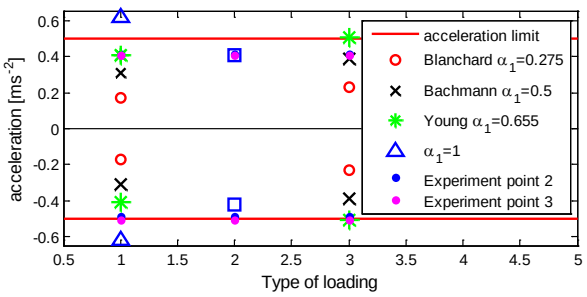

Fig 7 The summary of measured and computed results

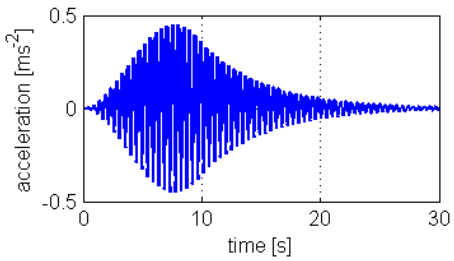

Fig 8 The acceleration of the beam midpoint Moving deterministic model

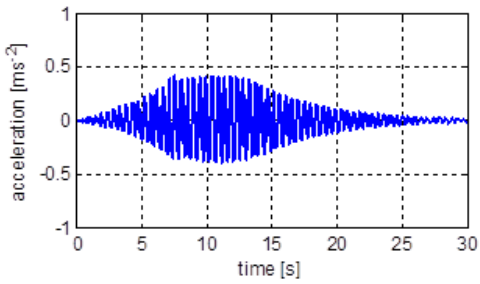

Fig 9 The acceleration of the beam midpoint Moving constant force

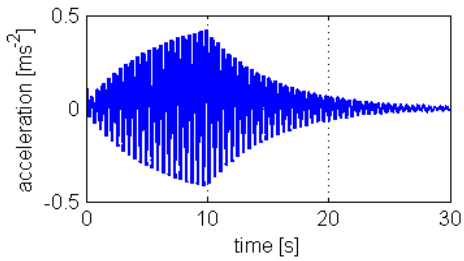

Fig 10 The acceleration of the beam midpoint Deterministic Model

6. CONCLUSION

The submitted paper is aimed at an analysis of the forced vibration of a simply supported beam loaded by synchronous pedestrians. Mostly the behaviour of pedestrians is described by the deterministic motionless force, which is considered in the point with maximal ordinate of appropriate mode of vibration. Therefore the alternative models for acting pedestrian were used in the study described in the paper. The obtained results were compared with the basic simple model. First of this models is the pulsating force moving along the structure. Secondly the moving constant force increased by dynamic increment was revolved. The results obtained from these three types of loading were compared with the in-situ experiment. The results of theoretical analysis and experiment are summarized in the *Tab. 1* respectively *Tab.2 and Tab.3* and at *Fig. 7 – Fig. 10*. The *Figure 7* shows us, that the best results, which we are able to obtained from the dynamical analysis, presented in this study, provides the Young's model in case of the moving deterministic model and the *Bachmann's* and *Ammann's* model in case of the motionless deterministic model placed at the mid-point of the structure. The moveable force with constant magnitude provides the very similar results as *Young's* and *Bachmann's* models.

Acknowledgment
This paper has been created thanks to the project (SGS 12/117/OHK1/2T/11) supported by the Czech Technical University in the Prague, which is gratefully acknowledged.

Literature

1. BACHMANN, H. and AMMANN, W. *Vibrations in structures – Induced by man and machines.* Structural engineering documents, International Association for bridge and structural engineering IABSE, Zürich, 1987. 165 p. ISBN 3-85748-052-X
2. Design of footbridges (Background document) (2008). *Human induced vibrations of steel structures*. RFS2-CT-2007-00033
3. FANFULOVÁ, E. *Experimental analysis of dynamic response of footbridges*. Prague, 2013. Bachelor thesis. Czech Technical University in Prague. Faculty of Civil Engineering
4. XUEWEI, CH., XIAOLEI H., CHEANG J. *The Research of Time-History Response Analysis of Floor Vibration Based on Simulation of Group Walking.* Proceedings of First International Conference on Modelling and Simulation. Nanjing. China. 2008. pp. 214-220. ISBN 978-1-84626-161-9.
5. ŽIVANOVIČ, S., PAVIČ, A., REYNOLDS, P. *Vibration serviceability of footbridges under human-induced excitation:* A literature review. Journal of Sound and Vibration. 2005. 279(1-2), 1-74. ISSN 0022460X