Singularities of a planar parallel 3-RTR mechanism – a screw theory approach

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Abstract The aim of this paper is to summarise the mathematical basics of the Screw theory and its application in mechanics. The paper provides methodology for finding the input-output equation for velocities of the parallel mechanism, that allows to obtain the velocities of the effector and of the actuated kinematic couples. This algorithm is applied to a 3 degrees of freedom planar parallel 3-RTR mechanism where we determine the input-output equation for velocities and based on this we investigate the singularities of the moving platform.

Keywords Distribution fitting, parameter estimation, goodness-of-fit test

1. INTRODUCTION

Computational methods in robotics play a significant role, primarily due to the constantly increasing complexity of robotic devices, requirements for the accuracy and stability of motion control, prediction of properties even in the design phase. Such tasks cannot be performed without detailed knowledge of the kinematics and dynamics of robotic devices, whether with a serial or parallel kinematic structure. Several proven approaches are currently used for the kinematic and dynamic analysis of mechanisms, such as the formulation of motion equations and the description of the kinematic structure based on Denavit-Hartenberg notation, using Euler angles, etc. Many of these methods have a very good application in mechanisms with a serial kinematic structure. On the other hand, for mechanisms with a parallel or hybrid structure, we are not always able to arrive at a reliable solution with these conventional approaches.

One of the methods for solving the kinematics of mechanisms is Screw theory. This theory provides mathematical tools for solving many tasks - from investigating the mobility of mechanisms, solving the direct and inverse problem for velocities, detecting singular positions to considering redundant actuators. The singularities influence many performances, including the workspace, dexterity, stiffness and load capacity of parallel robots. At singular configuration, the robot loses control over degrees of freedom. Either the robot gains one or more unexpected degrees of freedom what causes the degradation of natural stiffness and decrease of the load capacity in the direction of the additional degree of freedom, or the robot lies at a dead point where it is not able to be controlled. [1] In this paper, we focus on investigating the singularities of the planar parallel 3-RTR mechanism with asymmetrical position of the actuators in the limbs.

The organization of the paper is as follows: in section 2, the basics of the Screw theory are summarized. Section 3 presents the application of the theory to the mechanics. In section 4, we continue in the mechanical applications with differential kinematics of parallel mechanisms. Section 5 provides the velocity equation and the singularity analysis of the planar parallel 3-RTR mechanism.

2. MATHEMATICAL CONCEPT OF SCREW THEORY

In this section, we clarify the basic concepts of the Screw theory mathematical apparatus. We define the concept of a screw as a dual vector, its characteristics, as well as operations with screws.

2.1 Screw, line vector, couple

A screw \$ is an ordered pair of vectors $\boldsymbol{s}, \boldsymbol{s^0} \in R^3$, written in the form of a dual vector

$\$ = (\boldsymbol{s}, \boldsymbol{s^0})$.

Vector $\boldsymbol{s}$ represents the direction of the screw axis. Screw pitch

$$h = \frac{\boldsymbol{s} \cdot \boldsymbol{s^0}}{\boldsymbol{s} \cdot \boldsymbol{s}}$$

is the numerical characteristic of the screw. Operation „·“ represents the scalar product of vectors. In special case, when $h = 0$, such a screw is called a line vector or also Plücker coordinates of a line. The dual part $\boldsymbol{s^0}$ represents the moment of the line to the origin of the coordinate system and is defined as the vector product of the direction vector $\boldsymbol{s}$ of the line and the position vector $\boldsymbol{r}$ of any point on the line

$\boldsymbol{s^0} = \boldsymbol{r} \times \boldsymbol{s}.$

The primary and dual parts of the line vector fulfil the orthogonality condition

$\boldsymbol{s} \cdot \boldsymbol{s^0} = 0.$

If the pitch $h = \infty$, such a screw is called a couple, denoted as $\$ = (\boldsymbol{0}, \boldsymbol{s})$, under the condition $\boldsymbol{s} \neq \boldsymbol{0}$. [2] A couple is thus a screw with an infinite pitch, which axis has a given direction, but with an arbitrary location in space.

2.2 Operations with screws

On the set of screw one can define the following operations:
Let $\$ = (\boldsymbol{s}, \boldsymbol{s^0}), \$_1 = (\boldsymbol{s_1}, \boldsymbol{s_1^0}),\ \$_2 = (\boldsymbol{s_2}, \boldsymbol{s_2^0})$ be two arbitrary screw and let $\lambda \in R$. Then

The sum of screws [3]

$\$_1 + \$_2 = (\boldsymbol{s_1}, \boldsymbol{s_1^0}) + (\boldsymbol{s_2}, \boldsymbol{s_2^0}) = (\boldsymbol{s_1} + \boldsymbol{s_2}, \boldsymbol{s_1^0} + \boldsymbol{s_2^0})$

The product of a screw and a scalar [3]

$\lambda\$ = \lambda(\boldsymbol{s}, \boldsymbol{s^0}) = (\lambda\boldsymbol{s}, \lambda\, \boldsymbol{s^0})$

The results of both these operations are screws, too.

Alongside the mentioned operations, we define also the reciprocal product of screws. Let $\$_1 = (\boldsymbol{s_1}, \boldsymbol{s_1^0}),\ \$_2 = (\boldsymbol{s_2}, \boldsymbol{s_2^0})$ be arbitrary screws. Then the operation „$\circ$“, defined as

$$\$_1 \circ \$_2 = (\boldsymbol{s_1}, \boldsymbol{s_1^0}) \circ (\boldsymbol{s_2}, \boldsymbol{s_2^0}) = \boldsymbol{s_1} \cdot \boldsymbol{s_2^0} + \boldsymbol{s_2} \cdot \boldsymbol{s_1^0}, \qquad (1)$$

is called the reciprocal product of screws. [2] The matrix form of the reciprocal product is as

$\$_1 \circ \$_2 = \$_1\, \Delta\, \$_2^T,$

where Δ is a 6x6 square matrix in the form

$$\Delta = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}. \qquad (2)$$

Matrix $\boldsymbol{I}$ je a unit matrix, $\boldsymbol{0}$ is the null matrix, both of dimension 3x3. The result of the reciprocal product is a scalar. Two screws $\$_1 = (\boldsymbol{s_1}, \boldsymbol{s_1^0}),\ \$_2 = (\boldsymbol{s_2}, \boldsymbol{s_2^0})$ are called reciprocal if

$\$_1 \circ \$_2 = 0.$

Let us consider a screw in the form $\$ = (l, m, n|\, p, q, r)$. The coordinates of the primary and the secondary part of the reciprocal screw $\$^r = (\bar{l}, \bar{m}, \bar{n}|\, \bar{p}, \bar{q}, \bar{r})$ is obtained from the equation

$$p.\bar{l} + q.\bar{m} + r.\bar{n} + l.\bar{p} + m.\bar{q} + n.\bar{r} = 0. \qquad (3)$$

Since we have six unknown coordinates and only one equation, there will exist ∞^5 solutions. However, it makes sense to consider only linearly independent solutions, which reduces the number to five screws. In addition, for reciprocal screws are unit screws, i.e. their primary part is a unit vector. For a simpler interpretation of the found reciprocal screws, the reciprocity conditions of some pairs of the screws:

- Two line vectors are reciprocal if and only if they lie in a plane.
- Two couples are always reciprocal.
- A line vector and a couple are reciprocal if and only if they are orthogonal. [2]

2.3 Linear (in)dependency of screws

Similarly as with vectors, there can be defined a linear dependency/independency of screws since they are dual vectors. The screws $\$_1(\boldsymbol{s_1}, \boldsymbol{s_1^0}), \$_2 = (\boldsymbol{s_2}, \boldsymbol{s_2^0}),\ \ldots, \$_n = (\boldsymbol{s_n}, \boldsymbol{s_n^0})$ are said to be linearly independent if

$$\begin{aligned} c_1\$_1 + c_2\$_2 + \cdots + c_n\$_n \\ &= c_1(\boldsymbol{s_1}, \boldsymbol{s_1^0}) + c_2(\boldsymbol{s_2}, \boldsymbol{s_2^0}) + \cdots + c_n(\boldsymbol{s_n}, \boldsymbol{s_n^0}) \\ &= (\boldsymbol{0}, \boldsymbol{0}) \end{aligned}$$

and the real coefficients $c_1, c_2, \ldots, c_n$ are all equal to zero. Otherwise, the screws are said to be linearly dependent. With respect to the fact that the screw as a dual vector has six coordinates, in the 3D space there exist at most six linearly independent screws. [2] When solving the problem of linear (in)dependency of the screws, we proceed in the same way as for vectors.

3. SCREW THEORY IN MECHANICS

Any change in the position of a rigid body in space can be achieved by rotating the body around an axis and then moving it in the direction of the given axis. When these two movements are carried out simultaneously, it is nothing more than the movement of the body along a path in the shape of a helix. [4] In this section, we define the motion and force screws, which allow us to connect the concept of screw from the previous section with the concepts of speed, force and moment of forces. We also focus on the respective kinematic pairs expressed by means of screws.

3.1 Motion screw

Let us consider a rigid body that rotates in the space with an angular velocity ω around the axis defined by the direction vector $\boldsymbol{s}$. If we express the rotation axis as a line vector, then it is possible to describe the rotation of the body by an angular velocity line vector as

$$\omega\$ = \omega(\boldsymbol{s}, \boldsymbol{s^0}) = (\boldsymbol{\omega}, \omega\boldsymbol{s^0}). \qquad (4)$$

The dual part of the line vector (4)

$\omega\boldsymbol{s_0} = \omega(\boldsymbol{r} \times \boldsymbol{s}) = \boldsymbol{r} \times (\omega\boldsymbol{s}) = \boldsymbol{r} \times \boldsymbol{\omega} = \boldsymbol{v_0},$

represents the velocity of a point coincident with the origin. Thus, one can express the rotational motion of a rigid body by a motion screw, called twist, in the form

$\omega\$ = (\boldsymbol{\omega}, \boldsymbol{v_0}).$

If the rotation axis passes through the origin, the form of the twist is $\omega\$ = (\boldsymbol{\omega}, \boldsymbol{0})$ since the location vector $\boldsymbol{r}$ is zero vector. [2]

Let us consider the rigid body, that moves with a translational velocity v in the direction of $\boldsymbol{s}$. The vector of instantaneous translational velocity is given as

$\boldsymbol{v} = v\boldsymbol{s}.$

During the translation, each point of the rigid body draws the same trajectory. This means, that if we move the vector $\boldsymbol{s}$ parallel to its original placement, the velocity vector $\boldsymbol{v}$ does not change. In Screw theory, this can be expressed as a couple $\$ = (\boldsymbol{0}, \boldsymbol{s})$. Therefore, the translation of the rigid body as a state can be described by a screw

$$v\$ = v(\boldsymbol{0}, \boldsymbol{s}) = (\boldsymbol{0}, \boldsymbol{v}). \qquad (5)$$

The translation can be regarded as a rotation around the axis orthogonal to $\boldsymbol{s}$, that lies in infinity. [2]

General motion then corresponds to a sum of translational and rotational motion. Expressed as screws using (4) and (5) we gain the motion state of the rigid body as

$\omega\$ + v\$ = (\boldsymbol{\omega}, \omega\boldsymbol{s^0}) + (\boldsymbol{0}, v\boldsymbol{s}) = (\boldsymbol{\omega}, \omega\boldsymbol{s^0} + v\boldsymbol{s}).$

The pitch of the twist

$h = \dfrac{v}{\omega}$

is given as a ratio of translational velocity of the rigid body moving in the direction of the screw axis and the angular velocity of the body rotating around this axis. [5]

3.2 Force screw

Similarly, we can unify the force and the moment of forces into one common expression employing the screw.

Let the force f acts on a rigid body in the direction of vector $\boldsymbol{s}$, then the force vector can be written in the form of product $f\boldsymbol{s}$. Simultaneously, force f causes the presence of the moment of forces $\boldsymbol{M_0}$ about the origin in the rigid body such as

$$\boldsymbol{M_0} = \boldsymbol{r} \times \boldsymbol{f} = \boldsymbol{r} \times f\boldsymbol{s} = f(\boldsymbol{r} \times \boldsymbol{s}) = f\boldsymbol{s^0}.$$

The force acting on a body in the direction of $\boldsymbol{s}$ is given as a screw

$$(\boldsymbol{f}, \boldsymbol{M_0}) = (f\boldsymbol{s}, f\boldsymbol{s^0}) = f(\boldsymbol{s}, \boldsymbol{s^0}) = f\$.$$

Now let us consider that there are two parallel equally large, oppositely oriented forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ acting on a body. The effect of these forces creates a moment of the pair of forces $\boldsymbol{M}$, which acts in the direction of the vector $\boldsymbol{s}$ perpendicular to the plane of the acting forces $\boldsymbol{f}_1, \boldsymbol{f}_2$. The action of a pair of forces can be written as the screw $M(\boldsymbol{0}, \boldsymbol{s})$.

If a system of forces and moments of forces acts on a rigid body, we can write it down as the action of the resulting force and the resulting moment of forces using a wrench in the form

$$f\$ = f(\boldsymbol{s}, \boldsymbol{s^0}) = (\boldsymbol{f}, \boldsymbol{M^0}) = f(\boldsymbol{s}, \boldsymbol{s_0} + h_f\boldsymbol{s}),$$

where h_f is the wrench pitch – a ratio of the moment of the forces acting in the direction of the screw and resulting force. [5]

3.3 Kinematics of pairs

To determine the mobility of a mechanism using screws, we use the reciprocal product of screws (1). The reciprocal product of the force and motion screw represents the instantaneous work due to the force acting on the moving body. However, the coupling forces (moments of forces) do not induce any work when acting on the body, i.e. the reciprocal product of such a screw with a motion screw is zero. Regardless of the magnitude of the force acting in this way, the state of motion of the body does not change. If the screw $ determines the free movement of the body, then the reciprocal screw $\$^r$ represents the constraint (force or moment) with respect to the removed degree of freedom. Conversely, if $ represents a constraint, then $\$^r$ represents the movement that the constraint allows the body. [2]

Let us now consider a rigid body that is connected by a geometric constraint to a mechanism, and this constraint removes j degrees of freedom from the body, $j<6$. The set of all linearly independent motion screws describing the movements that the given constraint allows for the body is called the motion system of the body S_m. The set of all linearly independent screws that are simultaneously reciprocal to all screws of the motion system S_m is called the reciprocal force system S^R. It is true that the sum of free movements and constraints is always six, therefore

$$\dim(S_m) + \dim(S^R) = 6.$$

To each reciprocal force system S^R we can assign a constrained motion system S_m^C, which contains in a screw expression the movements limited by the given geometric constraint. For the coupling motion screws bound to the coupling force screws, the following holds: If the force screw is $\$^r = (\boldsymbol{s}, \boldsymbol{s_0} + h\boldsymbol{s})$, then the motion screw expressing the motion limited by the $\$^r$ constraint is given by [5]

$$\$_m^C = \left(\boldsymbol{s}, \boldsymbol{s_0} + \tfrac{1}{h}\boldsymbol{s}\right).$$

In the following section, we will describe the individual types of body constraints in space using motion screws and constraints that are in the form of reciprocal screws to them. For simplicity, we will only list constraints with one degree of freedom.

Let us consider two members connected by a rotational joint. We define a cartesian coordinate system according to fig. 1.

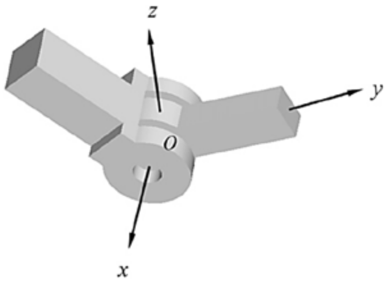

Fig. 1. Cartesian coordinate system for the rotational joint. [5]

The x-axis corresponds to the joint axis. Then the motion screw expressing the free motion is given by

$$\$ = (1, 0, 0 | 0, 0, 0).$$

For two members connected by a prismatic joint, we define the cartesian coordinate system according to fig. 2. The direction of displacement is defined in the direction of the x-axis. Then the motion screw expressing the free motion is given by

$$\$ = (0, 0, 0 | 1, 0, 0).$$

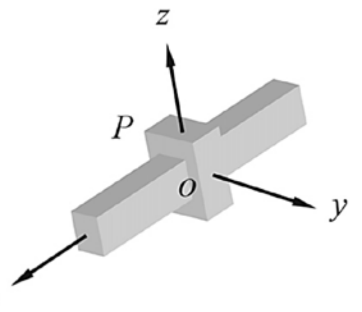

Fig. 2. Cartesian coordinate system for the prismatic joint. [5]

Now let us consider two members connected by a screw connection with a screw pitch h. We define the cartesian coordinate system according to fig. 3. The x-axis corresponds to the screw axis. Then the motion screw expressing the free movement is given by

$$\$ = (1, 0, 0 | h, 0, 0).$$

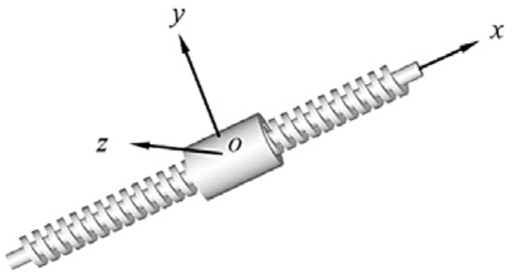

Fig. 3. Cartesian coordinate system for a helical joint. [5]

Other joints can be expressed as combination of the mentioned joints. Therefore, their screw expressions are combinations of the defined screws.

4. KINEMATIC ANALYSIS

An integral part of the analysis of the mechanism is its kinematic model, which is necessary both for the solution of its dynamics, as well as for its control and simulation. The task of kinematic analysis is to define the course of the position and orientation of the end effector over time, its speed and acceleration, or other higher derivatives of the position. Within kinematics, we distinguish four basic tasks:

- direct problem for location,
- inverse problem for position,
- direct problem for velocities,
- inverse problem for velocities.

4.1 The velocity equation

Let us consider the open kinematic chain that consists of the members, denoted as $j, j+1, j+2, \dots, m-2, m-1, m$. These members are joint together with helical pair where ${}^{k}\$^{k+1}$ denotes the helical pair between members k and $k+1$. Then the velocity of member m with respect to member j is given as

$$ {}^{j}\boldsymbol{V}^{m} = \omega_{j,j+1}\,{}^{j}\$^{j+1} + \omega_{j+1,j+2}\,{}^{j+1}\$^{j+2} + \dots + \omega_{m-1,m}\,{}^{m-1}\$^{m}. \qquad (6) $$

If the j-th member is the base of the serial robot and the m-th member is the end effector, one gains the equation for relative velocity of the end effector in the screw form. [3] At the same time, we get the solution of the direct tasks for speeds. The resulting velocity is found based on one configuration of the robot and the relative velocities of its members. [6] Writing the equation (6) in the matrix form

$$ \left({}^{0}\boldsymbol{V}^{n}\right)^{\boldsymbol{T}} = \boldsymbol{J}.\boldsymbol{\Omega}, \qquad (7) $$

where

$$ \boldsymbol{J} = \left(\left({}^{0}\$^{1}\right)^{T} \quad \left({}^{1}\$^{2}\right)^{T} \quad \cdots \quad \left({}^{n-1}\$^{n}\right)^{T}\right) \qquad (8) $$

is the Jacobi matrix (Jacobian) and

$$ \boldsymbol{\Omega} = \left(\omega_{0,1}\ \omega_{1,2}\ \omega_{2,3} \dots \omega_{n-1,n}\right)^{\mathrm{T}} \qquad (9) $$

the vector of relative velocities, we can find the solution of the inverse problem for velocities, as well. Multiplying (7) from left with an inverse $\boldsymbol{J}^{-1}$ leads to the vector of relative velocities (9) in the form

$$ \boldsymbol{\Omega} = \boldsymbol{J}^{-1}\left({}^{0}\boldsymbol{V}^{n}\right)^{\boldsymbol{T}}. $$

In the case of a parallel mechanism, the relative velocity of the moving platform as the end effector of the mechanism must be the same, regardless of the limb that is used to get the equation (6). The main problem lies within the fact that not all the kinematic pairs in the parallel mechanism are controlled. The solution is to eliminate the passive kinematic pairs in (6) by reciprocal multiplication. Repeating the process of multiplying (6) with the screw reciprocal simultaneously to all the screws of the passive joints in the limb through all the limbs of the mechanism leads to a system of equations, given in matrix form

$$ \boldsymbol{A\Delta V^{p}} = \boldsymbol{B\Omega_{a}}, \qquad (10) $$

where

$$ \boldsymbol{A} = \left[\$_1^R\ \$_2^R\ \$_3^R\ \$_4^R\ \$_5^R\ \$_6^R\right] \qquad (11) $$

is the matrix of screws reciprocal to screws of passive joints in respective limbs. $\boldsymbol{\Delta}$ is the operator of polarity defined in (2),

$$ \boldsymbol{B} = diag\left[\$_1^R \circ \$_{a_1},\ \$_2^R \circ \$_{a_2},\ \$_3^R \circ \$_{a_3},\ \$_4^R \circ \$_{a_4},\ \$_5^R \circ \$_{a_5},\ \$_6^R \circ \$_{a_6}\right] \qquad (12) $$

is the diagonal matric of coefficients and

$$ \boldsymbol{\Omega_a} = \left[\omega_{a_1},\ \omega_{a_2},\ \omega_{a_3},\ \omega_{a_4},\ \omega_{a_5},\ \omega_{a_6}\right]^{T} \qquad (13) $$

is the vector of relative velocities of the actuated kinematic pairs in respective limbs (here, (11), (12), (13) are expressed as for the parallel mechanism with six actuators). [3, 7, 8] If there exists the inverse of the matrix $\boldsymbol{A\Delta}$, then multiplying (10) with $(\boldsymbol{A\Delta})^{-1}$ we obtain the solution of the direct problem for velocities. On the other hand, if there exists the inverse of the matrix $\boldsymbol{B}$, then multiplying (10) with $\boldsymbol{B}^{-1}$ provides us with the solution of the inverse problem for velocities.

4.2 Singularities of a mechanism

Singularities of the serial mechanisms are identified via the Jacobi matrix (8) from the velocity equation (7). The serial mechanism is in a singular position when the Jacobi matrix is singular

$$ \det(\boldsymbol{J}) = 0, $$

or

$$ \det(\boldsymbol{J^T J}) = 0 $$

in case the matrix $\boldsymbol{J}$ is not square. For parallel mechanisms we determine the singular positions from (10). Three types of singularities can be distinguished

- 1st type singularity - matrix **B** is singular,
- 2nd type singularity - matrix $\boldsymbol{A\Delta}$ is singular,
- 3rd type singularity - both matrices $\boldsymbol{A\Delta}$ and **B** are singular. [3, 7]

5. KINEMATIC ANALYSIS OF THE 3 DOF PLANAR PARALLEL MECHANISM

Let us consider the planar parallel mechanism where the moving platform is connected to the base by three serial chains of type RTR (fig. 4).

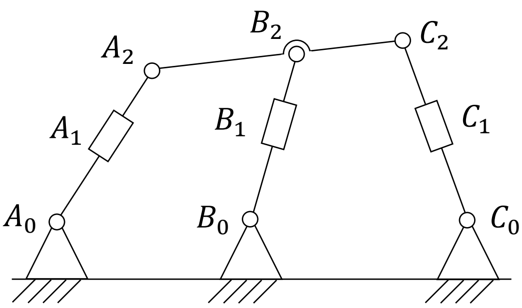

Fig. 4. The 3-RTR planar parallel mechanism.

The position of the coordinate system (fig. 5) is as follows: the mechanism lies in the plane xz, the axis y is perpendicular to the xz plane, oriented in the sense of right-hand rule.

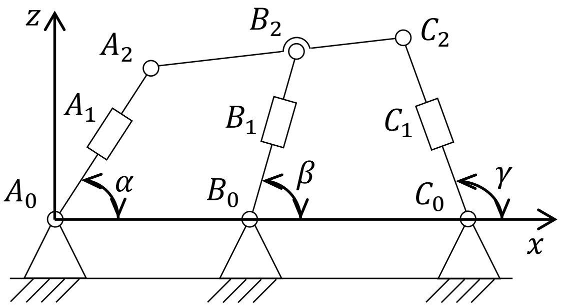

Fig. 5. The coordinate system of the mechanism.

The angles are depicted in fig. 5. The lengths of respective members are

$$|A_0A_1| = L_1 \quad |A_0A_2| = L_2 \quad |A_0B_0| = L_3$$
$$|B_0B_1| = L_4 \quad |B_0B_2| = L_5 \quad |B_0C_0| = L_6$$
$$|C_0C_1| = L_7 \quad |C_0C_2| = L_8$$

The mechanism has 3 degrees of freedom - a rotation around the axis parallel to the axis y, a translation in the direction of the axis x and a translation in the direction the axis z. According to this, the twist of the moving platform is of the form

$$\$_m = (0, \omega_y, 0 | \, v_x, 0, v_z \,).$$

Velocity of the moving platform is then expressed by (6) as

$$\begin{aligned}(0, \omega_y, 0 | \, v_x, 0, v_z \,) &= \omega_{A_0}\$_{A_0} + v_{A_1}\$_{A_1} + \omega_{A_2}\$_{A_2},\\ (0, \omega_y, 0 | \, v_x, 0, v_z \,) &= \omega_{B_0}\$_{B_0} + v_{B_1}\$_{B_1} + \omega_{B_2}\$_{B_2},\\ (0, \omega_y, 0 | \, v_x, 0, v_z \,) &= \omega_{C_0}\$_{C_0} + v_{C_1}\$_{C_1} + \omega_{C_2}\$_{C_2}.\end{aligned} \qquad (14)$$

Now we find the screws corresponding to the joints in the limbs. The revolute pairs are expressed as line vectors and the prismatic pairs are expressed as couples. The screws of the kinematic pairs are presented in tab. 1.

Tab. 1. The screws of the kinematic pairs in the mechanism.

	Screws
Chain $A_0A_1A_2$	$\$_{A_0} = (0\ 1\ 0|0\ 0\ 0)$, $\$_{A_1} = (0\ 0\ 0|L_1 \cos\alpha, 0, L_1 \sin\alpha)$, $\$_{A_2} = (0\ 1\ 0|-L_2 \sin\alpha, 0, \quad L_2 \cos\alpha)$.
Chain $B_0B_1B_2$	$\$_{B_0} = (0\ 1\ 0|0\ 0\ L_3)$, $\$_{B_1} = (0\ 0\ 0|L_4 \cos\beta, 0, L_4 \sin\beta)$, $\$_{B_2} = (0\ 1\ 0|-L_5 \sin\beta, 0, \quad L_3 + L_5 \cos\beta)$.
Chain $C_0C_1C_2$	$\$_{C_0} = (0\ 1\ 0|0, 0, L_3 + L_6)$, $\$_{C_1} = (0\ 0\ 0|L_7 \cos\gamma, 0, L_7 \sin\gamma)$, $\$_{C_2} = (0\ 1\ 0|-L_8 \sin\beta, 0, \quad L_3 + L_6 + L_8 \cos\gamma)$.

Three degrees of freedom assume the presence of three actuated kinematic pairs in the mechanism, one in each limb.
First, let us consider the location of the actuators in each chain as follows:

- In the chain $A_0A_1A_2$, it is the prismatic pair A_1.
- In the chain $B_0B_1B_2$, it is the revolute pair B_0.
- In the chain $C_0C_1C_2$, it is the prismatic pair C_1.

Now we must find the screws reciprocal to passive screws in the serial chains to transform (14) into the form of the input-output equation (10). For this purpose, we use (3) what leads to a system of two equations with six unknows for each chain. In the $A_0A_1A_2$ chain, it is

$$(t \cos\alpha, u, t \sin\alpha \,|\, v, 0, w \,). \qquad (15)$$

Writing the reciprocal screw (15) as a sum of screws for separate parameters,

$$t \cdot (\cos\alpha, 0, \sin\alpha \,|\, 0, 0, 0\,) + u \cdot (0, 1, 0 | \, 0, 0, 0) + v \cdot (0, 0, 0 | \, 1, 0, 0\,) + w \cdot (0, 0, 0 | \, 0, 0, 1\,)$$

we omit those screws that are reciprocal to every screw in the chain, the active included. The screw reciprocal to passive pairs in the $A_0A_1A_2$ chain is

$$(\cos\alpha, 0, \sin\alpha \,|\, 0, 0, 0\,). \qquad (16)$$

In the same manner, we would find the reciprocal screw in the $B_0B_1B_2$ chain

$$(\sin\beta, 0, -\cos\beta \,|\, 0, L_5 + L_3 \cos\beta, 0\,) \qquad (17)$$

and in the $C_0C_1C_2$ chain

$$(\cos\gamma, 0, \sin\gamma \,|\, 0, -(L_3 + L_6) \sin\gamma, 0\,). \qquad (18)$$

Multiplying (14) with reciprocal screws in the order – the first equation of (14) by (16), the second equation by (17) and the third one by (18), we obtain the equation (10) in the form

$$\begin{pmatrix} 0 & \cos\alpha & \sin\alpha \\ L_5 + L_3 \cos\beta & \sin\beta & -\cos\beta \\ -(L_3 + L_6) \sin\gamma & \cos\gamma & \sin\gamma \end{pmatrix} \begin{pmatrix} \omega_y \\ v_x \\ v_z \end{pmatrix} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_5 & 0 \\ 0 & 0 & L_7 \end{pmatrix} \begin{pmatrix} v_{A_1} \\ \omega_{B_0} \\ v_{C_1} \end{pmatrix}. \qquad (19)$$

From (19) after multiplication with the inverse of the matrix of coefficients on the left-hand side, we gain the velocity of the moving platform for the instantaneous configuration and velocities of the actuated pairs. On the other hand, multiplying (19) with the inverse of the matrix of coefficients on the right-hand side allows us to solve the inverse problem for velocities, i.e. the velocities of the actuated kinematic pairs knowing the velocity of the moving platform and the configuration of the mechanism at that moment.

Furthermore, we use equation (19) to identify singular positions of the parallel mechanism. The mechanism is in a singular position of the 1st type when

$$\begin{vmatrix} L_1 & 0 & 0 \\ 0 & L_5 & 0 \\ 0 & 0 & L_7 \end{vmatrix} = 0.$$

Since the lengths $L_1, L_5, L_7 \neq 0$, the mechanism cannot reach the 1st type singular position. This means that the moving platform is not able to move when the actuators are stopped. The mechanism is in a singular position of the 2nd type when

$$\begin{vmatrix} 0 & \cos\alpha & \sin\alpha \\ L_5 + L_3 \cos\beta & \sin\beta & -\cos\beta \\ -(L_3 + L_6) \sin\gamma & \cos\gamma & \sin\gamma \end{vmatrix} = 0,$$

or

$$L_5 \sin(\alpha - \gamma) + L_6 \sin\gamma \cos(\beta - \alpha) + L_3 \sin\alpha \cos(\beta - \gamma) = 0. \qquad (20)$$

Reaching the 2nd type singularity position requires such configuration when (20) will be fulfilled.

Now we model the configuration where the actuators are located symmetrically, as follows:

- In chain $A_0A_1A_2$, it is the revolute pair A_0.
- In chain $B_0B_1B_2$, it is the revolute pair B_0.
- In chain $C_0C_1C_2$, it is the revolute pair C_0.

Here, the screw reciprocal to passive pairs in the $A_0A_1A_2$ chain is

$(-\sin\alpha, 0, \cos\alpha \mid 0, -L_2, 0)$. (21)

For the chain $B_0B_1B_2$, it is the screw

$(\sin\beta, 0, -\cos\beta \mid 0, L_5 + L_3\cos\beta, 0)$ (22)

and for the chain $C_0C_1C_2$, it is

$(\sin\gamma, 0, -\cos\gamma \mid 0, L_8 + (L_3 + L_6)\cos\gamma, 0)$. (23)

Again, we multiply the system (14) with reciprocal screws in the order - the first equation of (14) by (21), the second one by (22) and the third one by (23). This way we obtain the equation (10) in the form

$$\begin{pmatrix} L_2 & \sin\alpha & -\cos\alpha \\ L_5 + L_3\cos\beta & \sin\beta & -\cos\beta \\ L_8 + (L_3 + L_6)\cos\gamma & \sin\gamma & -\cos\gamma \end{pmatrix} \begin{pmatrix} \omega_y \\ v_x \\ v_z \end{pmatrix} =$$
$$= \begin{pmatrix} L_2 & 0 & 0 \\ 0 & L_5 + L_3\cos\beta & 0 \\ 0 & 0 & L_8 + (L_3 + L_6)\cos\gamma \end{pmatrix} \begin{pmatrix} v_{A_1} \\ \omega_{B_0} \\ v_{C_1} \end{pmatrix}.$$

The mechanism is in a singular position of the 1st type when

$$\begin{vmatrix} L_2 & 0 & 0 \\ 0 & L_5 + L_3\cos\beta & 0 \\ 0 & 0 & L_8 + (L_3 + L_6)\cos\gamma \end{vmatrix} = 0. \quad (24)$$

Even if $\beta = \gamma = \frac{\pi}{2}$, the determinant (24) will not be equal to 0 because lengths $L_2, L_5, L_8 \neq 0$. Therefore, the mechanism will never reach the 1st type singular position. However, the mechanism can reach the singular position of the 2nd type when $\alpha = \beta = \gamma$. Then

$$\begin{vmatrix} L_2 & \sin\alpha & -\cos\alpha \\ L_5 + L_3\cos\beta & \sin\beta & -\cos\beta \\ L_8 + (L_3 + L_6)\cos\gamma & \sin\gamma & -\cos\gamma \end{vmatrix} = 0.$$

This can be prevented if we do not have the base and the moving platform of the same length. Despite this, the mechanism can be in a singular position when $\alpha = \beta = \gamma = 0$.

Again, we consider the configuration with symmetrical location of the actuators, this time in the prismatic pairs. The screws reciprocal to passive screws in the limbs are already known for the limb $A_0A_1A_2$ and $C_0C_1C_2$ – these are the screws (16) and (18), respectively. The reciprocal screw for the limb $B_0B_1B_2$ is of the form

$(\cos\beta, 0, \sin\beta \mid 0, -L_3\sin\beta, 0)$.

Applying the reciprocal screws on the (14), we obtain the equation

$$\begin{pmatrix} 0 & \cos\alpha & \sin\alpha \\ -L_3\sin\beta & \cos\beta & \sin\beta \\ -(L_3 + L_6)\sin\gamma & \cos\gamma & \sin\gamma \end{pmatrix} \begin{pmatrix} \omega_y \\ v_x \\ v_z \end{pmatrix} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_4 & 0 \\ 0 & 0 & L_7 \end{pmatrix} \begin{pmatrix} v_{A_1} \\ v_{B_1} \\ v_{C_1} \end{pmatrix}.$$

The mechanism will not reach the 1st type singular position since the lengths $L_2, L_4, L_8 \neq 0$. Also, it will not reach the 2nd type singular position when the angles α, β, γ do not have the same value.

6. CONCLUSION

In the paper, we summarised the basic mathematical concepts of the Screw theory. We have shown the connection of this mathematical apparatus with mechanics and presented a known algorithm for finding an input-output velocity equation for parallel mechanisms. Subsequently, we applied it in solving the kinematic analysis of a planar parallel mechanism with three limbs of the RTR type. We have found the input-output form of the equation for the velocities and conditions under which the mechanism will reach a singular position inside its working space.

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